CHAPTER 2

Structure of Crystal, Bucket of Dust

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1. The Beauty of the Inert-Dead

Every mathematical argument tells a story. But where is that story located? Do the chapters open in Plato's heaven, outside time, outside the cave of mere human projection? Is the true story of mathematics something so far beyond spelunking materiality that intuitions and mere images must be left behind? Or are these stories precisely ones of things and forces, surfaces and movement?

To address these questions about mathematical narration, I want to focus on the "geometrodynamic" vision of that school-founding, profound, quirky, creative, and provocative American physicist, John Archibald Wheeler. Far less known than many of his contemporaries such as J. Robert Oppenheimer, Hans Bethe, or Niels Bohr, or his student, Richard Feynman, Wheeler nonetheless had an immense effect on mathematical physics. He wrote the first important paper on the theory of nuclear fission; he introduced fundamental physical notions such as the S-matrix and the compound nucleus; he contributed powerfully to the understanding of stellar collapse: "black hole" is his term, Wheeler having seized on it and injected it into the mathematical lexicon after an anonymous member of the audience called it out during one of Wheeler's lectures. Wheeler produced the first theory of positronium; he played a crucial role in establishing the enormous plutonium reactor plant in Hanford, Washington, during World War II; at Princeton, where he taught for decades, he led one of the principal design teams for the hydrogen bomb; and he helped launch elementary particle physics as a major research field. But I will only tangentially be interested in the biographical. Instead, my aim is to characterize his way of renarrating mathematics as a kind of compound machine—from 1952 forward—as he sought to bring general relativity into the mainstream of physics.

To throw Wheeler's math-machines into relief, I want to contrast his way of thinking about mathematics with that of the famous French mathematical collective that came, in 1934–35, to call itself Bourbaki. Why compare the theoretical physicist Wheeler and the mathematical collective, Bourbaki? Because, in a certain sense, even though they were about as far apart as possible on the idiom spectrum that spans from formal-algebraic to informal-geometric, they ended up telling what are actually parallel stories. More proximately, both Wheeler and Bourbaki's first-generation members were born around 1908, plus or minus four or five years.[1] Wheeler came alive as a physicist during his 1934 trip to Bohr's Copenhagen Institute; Bourbaki came to existence around the same time in France. Both had a quite powerful effect on a generation of thought about the exact sciences; both came to believe there was a natural starting point to reasoning about mathematical structures somewhere in the region of set theory. Both wrote epochal treatises—Bourbaki's *Elements of Mathematics*, Wheeler's (with Charles Misner and Kip Thorne) *Gravitation*.[2] But similarities can deceive: these were two radically different pictures of mathematics. It is the *contrast* in their way of relating the narrative of mathematics that interests me here.

The Bourbaki members aimed their story of mathematics to be the non-narrative narrative, the account outside time, a structure, an architecture to be contemplated as it ordered "mathematic" from set theory on out. Wheeler's is, by contrast, a multipart device—his covariant derivatives are three-slot input-output machines; his is a world where instructions pull dimensionality itself out of a Borel set of points that Wheeler dubbed a "bucket of dust." Bourbaki's account is a crystal of symbols, Wheeler's a set of linked machine-stories, a hybrid of discovery accounts, speculative machine-like functions and mechanisms.

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For the Bourbaki collective of young French mathematicians who gathered around the École normale supérieure between the world wars, nothing was more important than clearing the congested reasoning of premodern mathematics. Their guiding elders had died in tragic number during the Great War, and by the time the first generation of Bourbaki came of age, they were looking not to the French antecedents so much as to the great modernizers of German abstract mathematics, such as B. L. Van der Waerden in his *Moderne Algebra* (1930–31).[3] In the mathematical collective's famous "Architecture of Mathematics" (1950),[4] though this is not so regularly noticed, the many-in-one mathematician invoked two governing metaphors of modernity, one from the imperious labor rationalizer Frederick Winslow Taylor, the other from the crusading urban rationalizer Baron Georges-Eugène Haussmann. Both spoke strikingly, in different ways, to the particularly, peculiarly Bourbakian narrative of modernity.

Bourbaki invoked the "economy of thought," physicist and philosopher Ernst Mach's insistent rallying cry that made simplicity of reasoning a criterion of choice among theories. Here in mathematics, Bourbaki insisted on it, too: " 'Structures are tools for the mathematician; as soon as he has recognized . . . relations which satisfy the axioms of a known type, he has at his disposal immediately the entire arsenal of general theorems which belong to the structures of that type. Previously . . . he was obliged to forge for himself the means of attack . . . their power depended on his personal talents and they were often loaded down . . . from the peculiarities of the problem that was being studied. One could say that the axiomatic method is nothing but the 'Taylor system' for mathematics."[5] For Taylor, the micro-examination of every idiosyncratic arm, head, or finger motion was key to cutting wasteful effort from production: economy of action. For Bourbaki, studying every mathematical move, step by step, could strip idiosyncratic effort in this most abstract of sciences: economy of thought.

Quick to dissociate themselves from the implication of machine-like reasoning, Bourbaki members were not mechanists. The machine metaphor was, for them, "a poor analogy . . . the mathematician does not work like a machine, nor as the workingman on a moving belt."[6] The mathematician, they urgently added, worked by "direct divination" in *advance* of experience rather than (as with the mere worker) on the basis of experience. Nevertheless, Bourbaki's initial Taylorism still stands: there are repetitive actions in mathematics, structures that occur again and again (the group structure, for example) and, just as Taylor had done for forging pig iron, Bourbaki wanted to gain the economy of action offered by seeing that one need not reinvent a process each time.

Individualism gone amok is a waste on the factory line *and* a waste on the chalkboard. Economy means streamlining, cutting excess movement—be it of hand or mind. It means focusing not on the content of mathematical objects but instead on *structures* that, according to Bourbaki, "can be applied to sets of elements whose nature has not been specified."[7] When Bourbaki tackled algebra, it sought to display the discipline's hierarchy of structures and so make evident its unity. Put aside as uninteresting the endless philosophical harkening after the meaning or reference of mathematical objects—so argued the mathematical modernizers. But put aside too the particularity of other paths into mathematics, not least the geometric. Robert Osserman, while not diminishing the power of the Bourbaki approach, pleaded for a very different style of mathematical reasoning: geometry "tends to be a rather ragged and uneven affair, full of loose ends, unfinished business, and decorative detail. It is a kind of antithesis to the neat, elegant, and rigid structure that may or may not succeed in housing and containing it."[8]

If Bourbaki drew its first modernizing metaphor from the factory line, it seized the second from the urban fabric of Paris itself. In words the authors of "Architecture" chose, "[Mathematics] is like a big city, whose outlying districts . . . encroach incessantly . . . while the center is rebuilt from time to time, each time in accordance with a more clearly conceived plan and a more majestic order, tearing down the old sections with their labyrinths of alleys, and projecting towards the periphery new avenues, more direct, broader and more commodious."[9] Here is a simile of high modernity, Bourbaki as Haussmann—that mid-nineteenth-century urban bulldozer who tore through the ramified ancient neighborhoods of Paris, sending open, radial avenues out from the core. Haussmann ploughed his avenue de la Grande Armée from the center to the edge of the city; Bourbaki would do this with deductive hierarchies of structures.

> [I]t cannot be denied that most of these [abstract forms, mathematical structures] had originally a very definite intuitive content; but, it is exactly by deliberately throwing out this content, that it has been possible to give these forms all the power which they were capable of displaying and to prepare them for new interpretations and for the development of their full power.[10]

Claude Chevalley, one of the founding members of Bourbaki back in December 1934, always insisted on the extraction of the mathematical structures from questions of origin or application. His daughter recalled his ascetic stance:

> Rigor consisted in getting rid of an accretion of superfluous details. Conversely, lack of rigor gave my father an impression of a proof where one was walking in mud, where one had to pick up some sort of filth in order to get ahead. Once that filth was taken away, one could get at the mathematical object, a sort of crystallized body whose essence is its structure. When that structure had been constructed, he would say it was an object which interested him, something to look at, to admire, perhaps to turn around, but certainly not to transform. For him, rigor in mathematics consisted in making a new object which could thereafter remain unchanged.
>
> The way my father worked, it seems that this was what counted most, this production of an object which then became inert-dead, really. It was no longer to be altered or transformed. Not that there was any negative connotation to this. But I must add that my father was probably the only member of Bourbaki who thought of mathematics as a way to put objects to death for esthetic reasons.[11]

Chevalley may have been (as his daughter suggested) a bit extreme, but the impulse to collect and codify rather than apply, to delimit rather than interpret, ran deep in the Bourbachiste project. Intriguingly, the self-narrative of the Bourbakians was in many instances highly historicized, a story of heroic modernization, exemplified by writing under the signs of Taylor and Haussmann, but also by their whole mode of existence as a depersonalized, collective author. But their mathematical story was one to be grasped, not developed through an inner sense of time unfolding.

As to the mathematics itself, read (if anyone ever did actually read) from the first volume of *Éléments de mathématique* straight through the last, the tomes stood for a vision of a concentrically layered structure, repetitive, cumulative, hermetic. Yet in a certain sense, reading as such, the sequential absorption, seems to pull against the Bourbakian ideal captured in the title, "Architecture of Mathematics." True, a building must stand the second floor on the first, the first on the foundation; but the completed edifice itself stands as a whole, not as a temporally developed sequence. If wishes were horses, one might think, we'd ride into this mathematics with a god's glance, not a human's walk.

We may "read" a building in various ways—through its history, through its historical allusions, through its engineering systems, or through its urban context. But in this human-made object there is no universally acknowledged time ordering. So it is with the Bourbakian project. Forget for a moment the later and even in some cases contemporary criticisms (endlessly repeated: insufficient Gödel, insufficient probability, insufficient mathematical physics . . .). As a partially realized vision of mathematics, here is a picture of a narrative outside time, a structure of structures voided not only of the physicality of objects but even of the specific, purely mathematical referentiality of mathematical entities. Here was supposed to be relations of relations to be contemplated out of time and out of space.

This meant, as Jean Delsarte argued at the very first meeting—Café Capoulade, noon, December 10, 1934—that one should begin expositions with the most general of statements, and only then proceed to the particular. For the treatise they originally had intended to write, this meant "there should be an abstract and axiomatic presentation of some essential general notions (such as a field, operation, set, group, etc.)." At first, the group called this opening to their story the "abstract package"; later, ambition expanded to embrace a full-on unification of mathematics: the "mother structures."[12]

2. Ingenious Things

How far can one get from the Café Capoulade? I would like here to give voice to a peculiarly American, Midwestern, machine picture of mathematics developed by John Archibald Wheeler. In many ways, as a mathematical narrator, Wheeler must be considered the epitome of everything Bourbaki disdained: a mathematical physicist who refracted both mathematical concepts and even mathematical demonstration through the idea of mere machines that had so shaped him growing up. Here is a vision of mathematics laden with intuitions, diagrams, machines: stories of discovery hard against exposition of technical matters. Wheeler was, quintessentially, the scientist who insistently cycled philosophical questions of meaning through the technical work. This is a mathematics set deep in the hewn-out limestone caves, far below the luminous forms of Plato's heaven.

Wheeler grew up on an American farm, not in the French capital; he spent his youngest years not in awe of pure knowledge but watching electricity being installed in the farmhouse and swiping dynamite caps from where they were stored in the pig barn.[13] Wheeler: "My father was very much interested in invention and in Yankee ingenuity, as it was called in those days."[14] Having worked as a librarian here and there, spending quite some time at Brown University's library, Wheeler's father—and Wheeler—gained a strong "idea of the needs and demands of a community like that—an industrial community. There was lots of silver working, brass working, machine shop work; and people all the time coming in to get answers to questions . . . a carryover of this ideal that we had in this country coming from England of science as connected with the common welfare, the university of the common people, a Cooper Union idea sort of thing."[15] Wheeler wired up telegraphs, built combination safes out of carved-wood parts, fashioned various kinds of guns, wired up radios, hammered together a functional calculating machine. With a young friend, he started a "gun and safe company."[16]

Wheeler's father took him to see the Waltham Watch Company outside Boston, where, the physicist remembered, "it was marvelous to see these little machines turning out parts and picking up parts and moving [them] from one place to another and assembling them."[17] Wheeler's words embraced a mechanical cornucopia, on the whole, highly sophisticated, but (radios excepted) not electrical. One book that Wheeler recalls poring over with deep affection was by Franklin D. Jones and bore the rather baroque title, *Mechanisms and Mechanical Movements: A Treatise on Different Types of Mechanisms and Various Methods of Transmitting, Controlling and Modifying Motion, to Secure Changes of Velocity, Direction, and Duration or Time of Action.* Issued by the Industrial Press in 1920, it aimed to teach designers and inventors and has been reprinted myriad times, as *Ingenious Mechanisms.*[18]

Jones made it clear on every page that machines were devices to transmit and alter motion—an understanding that goes back deep into the Victorian age. James Clerk Maxwell wrote extensively on this notion of the machine, developing a full-bore classification of devices for one of the great industrial fairs—but Maxwell was neither the first nor the last to take the machine in just this way.[19] Franklin Jones began this way: "The designers of machines or mechanisms in general are constantly engaged in the solution of problems pertaining to motion and its transmission. The motion derived from some source of power must be modified to produce certain effects, and various changes in regard to velocity, direction, and time of action may be necessary."[20] Not only did the author want to explain how motions can be produced and controlled, he aimed to do so in a way that securely bound the practical to the theoretical. Abstract theories alone, he insisted, "give an inadequate conception of their application in the design of mechanisms of various types."[21] All this had a special importance now (according to Jones) because of the increasing use of automatic machines in many branches of production—such as Wheeler had seen in the Waltham Watch Company.[22]

How, then, to modify energy to convey motion? One could proceed in the simplest cases by shafts, by links, by levers; by the more sophisticated means of a universal joint. Combining these building blocks could yield more elaborate machines, such as the pantograph, which could miniaturize motion (for example, from the movements of an engine head to the recording pen on an indicator card) (figure 2.1). In this way, with spiral, worm, or planetary gearing, with chains, belts, and cams, the recognizable ingenious machines could be understood and new ones designed—devices for changing and controlling speed, for converting from rotary to rectilinear motion and back, for reversing motion, for making it quick return or intermittent (for example, to build an adding machine). Jones's original volume culminated in automatic feeding mechanisms—one of the most sophisticated of early twentieth-century mechanical devices. Here were machines with inclined shoots and revolving magazines, others that fed screw blanks or (the book was copyrighted in 1918) bullet shells (figure 2.2).

Suppose, for example, you needed a device to sort your bullets so they reached the automated press tools pointy end first, regardless of their

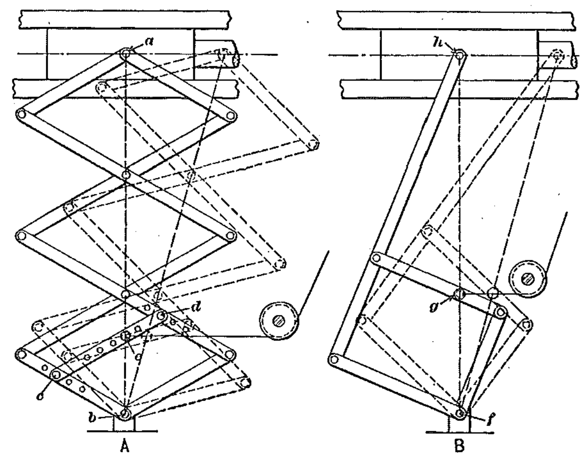

Figure 2.1. Pantograph diagrams. (From Jones, *Mechanisms and Mechanical Movements*, 1920, figure 13, p. 21).

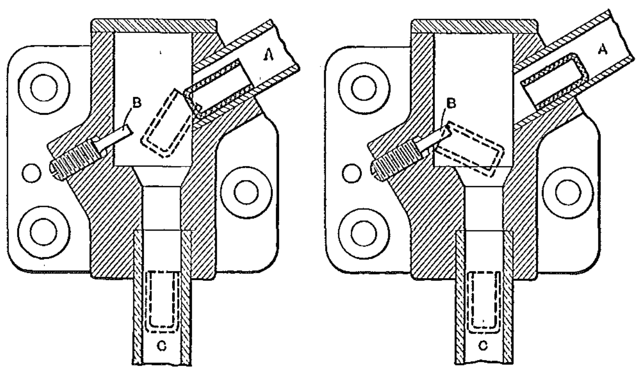

Figure 2.2. Automatic feed, orients shells. (From Jones, *Mechanisms and Mechanical Movements*, 1920, figure 6, p. 290).

original orientation (figure 2.3). This machine—like all the sophisticated ones—tells a little story, one that unfolds in time:

> [B]ullets enter the tube *A* which connects with a hopper located above the press. An "agitator tube" moves up and down through the mass of bullets in the hopper and the bullets which enter the agitator tube drop into tube *A*. As each bullet reaches the lower end of this tube, it is transferred by slide *C* (operated by cam *D*

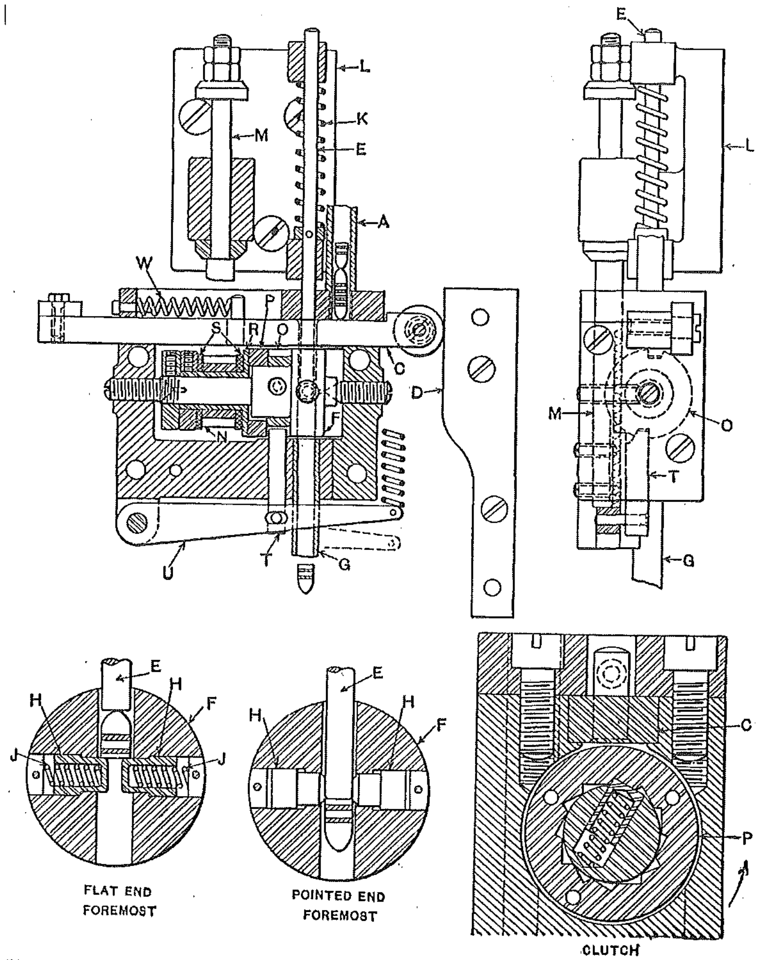

Figure 2.3. Feeding mechanism delivering bullets pointed ends foremost. (From Jones, *Mechanisms and Mechanical Movements*, 1920, figure 7, p. 292).

> attached to the cross-head) to a position under the rod *E*. The rod-holder *L* is also carried by the cross-head. Whenever a bullet enters tube *A* with the rounded or pointed end downward, it is simply pushed through a hole in dial *F* and into feed-pipe *G* leading to the dial feed-plate of the press. This feed-plate, in turn, conveys the bullets to the press tools where such operations as swaging or sizing are performed.[23]

Every machine account is a story like this one, a kind of picaresque novel, with a bundle of energy as hero. The shell moves; if it is upright it passes, if it is reversed it hits the protrusion and is flipped, then it falls out the shoot. . . . As in a story, the spotlight of our attention follows a thing or motion as it traverses obstacles, undergoes transformation over time, and emerges different than it was at the outset. Complex machines are assemblies of such stories. What I'm arguing here is that this assembly of machine stories is precisely how to understand Wheeler's mathematical physics—and ultimately how to understand his conception of the universe itself as a kind of mathematics machine, driven in the beginning by a logical array of propositions and evolving, through a long sequence of transforming theory machines, this time around, into the phenomenal world in which we live.

3. Creation, Annihilation, and the Universe Machines

Before 1952, Wheeler had no special interest in mathematics beyond what was needed for quantum physics. He spent a formative period with Niels Bohr in Copenhagen during 1934–35, a collaboration that continued throughout the later 1930s, culminating in their writing a joint theoretical paper on nuclear fission in 1939. Wheeler learned much from Bohr, not just about the content of nuclear physics and quantum mechanics but also about a way of proceeding. Always, Bohr looked for paradoxes as a way to enter more deeply into the physics—and Wheeler took this, as so much from Bohr, to heart.[24] During the war, Wheeler spent his time far from the hothouse environment of Los Alamos, working with engineers to scale up the reactors to produce industrial quantities of Pu239. Building on that experience, even before the war had come to a close, Wheeler began plotting and scheming to form a new field of elementary particle physics, large-scale accelerators, and an interdisciplinary team of scientists to tackle it. It was a vision of the new discipline (and its main laboratory) predicated directly and unambiguously (though obscured for security reasons) on the laboratories of Los Alamos, Hanford, and Oak Ridge, which by 1945 had made nuclear weapons production a larger industry than automobile manufacture.[25]

After hostilities ended, Wheeler went back to his old fascination with electrons as a guide to everything—a mission he abandoned when it became clear that electrons never traveled alone ("undressed") but always with the virtual particles that Feynman, Schwinger, and Tomonaga had uncovered. With the enthusiasm of a "reformed drunkard" (Wheeler's words), he threw himself into reverse. Having tried to dispense with fields in favor of particles, from 1947 forward he was after a field theory that would account for everything.[26]

Wheeler oscillated back and forth between machines and theory. In control of some of the captured V2 missiles launched from White Sands Proving Grounds, Wheeler organized some of these missions to probe cosmic rays. He was instrumental in helping to formulate the new interdisciplinary national laboratory that eventually became Brookhaven. And he was never far from defense matters and the devices they required. After the Soviets detonated their first nuclear weapon in August 1949, the hydrogen bomb rose, fiercely contested, to the height of American nuclear policy. In January 1950 President Truman approved the crash program; in January 1951, Stan Ulam and Edward Teller wrote their secret memo containing the guiding ideas for radiation implosion, the scheme that finally set the project on a definite path. Wheeler engaged immediately, running a design group that very practically undertook to simulate the explosion within the new designs: fission trigger, radiation-driven implosion, thermonuclear fusion.

> 29 March 1951. Dear Dick [Feynman]: I know you plan to spend next year in Brazil. I hope world conditions will permit. They may not. My personal rough guess is at least 40 percent chance of war by September, and you undoubtedly have your own probability estimate. You may be doing some thinking about what you will do if the emergency becomes acute. Will you consider the possibility of getting in behind a full scale program of thermonuclear work at Princeton through at least to September 1952? . . . Both Edward Teller and I would like to describe them [new work, undoubtedly focused on the novel "Teller-Ulam" design] to you in person to see if you don't think it is urgent for the defense of this country that most promising of these schemes be developed as soon as possible.[27]

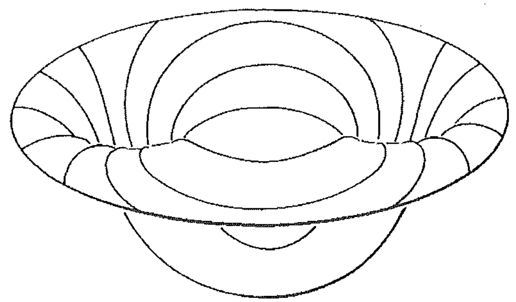

Figure 2.4. Wormholes. (From Misner, Thorne, and Wheeler, *Gravitation*, figure 44.1, p. 1200).

Feynman replied he was "uncomfortably aware of the very large chance I will be unable to go" to Brazil, but he did not want to commit to any project until events were clearer.[28] In the event, Feynman did go to Brazil, while Wheeler directed the Matterhorn B Project at Princeton that demonstrated, using computer simulations, that the new design would work. On Halloween 1952, the United States lit "Mike," the first H-bomb, a 10-megaton liquid-fueled monster, sending shock waves measurable by seismograph on the other side of the world and removing the South Pacific island of Elugelab from the face of the earth.

Back in Princeton, Wheeler began teaching general relativity—the first time the subject had been given as a course in Princeton's history. A few days after the Mike test, he set out some goals for the course in the first of his relativity notebooks. High on the list was this: "Want paradoxes as we go along."[29] That did it. From then on the project of a quantum-inflected general relativity became his golden fleece. His notebooks again, this from October 1953: "Be conservative; take q[uantum] m[echanics] and gravitation seriously down to very smallest distances."[30]

Around 1955, Wheeler began exploring ways in which one might imagine reconciling the still new quantum electrodynamics and gravity.[31] Quantum electrodynamics (QED) held that the vacuum was constantly seething virtual particles, for example pairs of electrons and positrons appearing and disappearing as allowed by the uncertainty principle. To avoid the breakdown of electrodynamics at the location of a point charge, Wheeler revived an old idea that this point location might actually be a multiply connected space with closed field lines (see figure 2.4). Quantum

electrodynamics would then make fluctuations into spontaneous fluctuations of the topology of space-time (new connecting handles arising and disappearing) at the Planck scale—to accommodate the short-lived virtual electrons and positrons.

As Wheeler pushed on mathematics to capture the conjoint project of general relativity and quantum physics, he, like Einstein before him, turned increasingly to mathematics. He needed additional mathematics to handle the Achilles' heel of the theory—the inability of geometrodynamics to address the problem of spin-½ particles (such as the neutrino or electron) within the frame of differential geometry. In 1966 he began organizing a conference with the express purpose of joining physicists and mathematicians in a common effort. The recruit he certainly wanted most to lure was Feynman, to whom Wheeler composed a long and detailed letter that included this:

> In the case of mathematics courses for physicists, the lecturer is not expected as a rule to go into specialized recent advances of the lecturer's own research: what most physicists need in order to acquire a rudimentary working knowledge of branches of mathematics that he has not yet manipulated is:
>
> — *intuition* (for example, an analogy with a familiar notion taken from the field of real numbers, the theory of complex variables, Riemannian geometry, etc.)
> — *motivation* (why one introduces a new sophisticated notion)
> — *examples* ("things" one does, giving enough apparatus to see how the "thing" works.
> . . . Some of us may be able to express ourselves in a language meaningful to mathematicians; others may have to ask for the forbearance of their colleagues for still talking "pidgin mathematics."[32]

Conciliatory as Wheeler may have wanted to be in allowing that physicists might speak "pidgin mathematics," this project interested Feynman not a whit. This was not because Feynman was indifferent to gravity; on the contrary, he had himself been involved with re-presenting Einsteinian gravity as a field theory in flat space. No, it was not that. In the 1960s (and indeed on through the mid-1980s), elementary particle physicists prided themselves on their *lack* of mathematical sophistication. Mathematics, it was taught—I was taught—was to physics a kind of cleanup squad that came after the parade had passed. Count on your fingers, learn some group theory, use path integrals with a devil-take-the-hindermost attitude toward well-definedness. This is what it really meant to do particle physics. Feynman epitomized the views of the proudly unmathematical theoretical physicist. Not one to mince words, he let his former thesis adviser know posthaste: "Dear John, I am not interested in what [carat: "todays'"] mathematicians find interesting. Sincerely yours, Dick."

In developing his machine-like way of narrating mathematical reasoning, Wheeler may have been in part responding to a long-standing suspicion toward mathematics among American physicists. Or, more simply, and probably more deeply, he may have been drawing on his own trajectory—one that had taken him through an intense engagement with machines, an original intention to become an engineer, and a lasting fascination with mechanisms and devices, from the calculator and radio through the vast national defense projects of the fission and fusion bomb.

A few examples from *Gravitation* give the idea:

"The 1-form is a machine to produce a number ('bongs of bell' as each successive integral surface is crossed) out of a displacement (approximation to a concept of a tangent vector)."[33] A 1-form is imagined as a sequence of sheets each of which sounds a bell when it is penetrated by the $\boldsymbol{v}$. So in this figure, $\boldsymbol{v}$ goes through four sheets and about halfway to the fifth—this makes $\langle \sigma, \boldsymbol{d} \rangle = 4.4$. Wheeler draws the diagram shown in figure 2.5.

Heading into electromagnetism—and slightly more complex—are 2-forms, to be imagined (*Gravitation* instructs) as oriented honeycomb structures formed by "wedge multiplication" from the two 1-forms $\boldsymbol{dy}$ and $\boldsymbol{dz}$: the electromagnetic field, $\boldsymbol{F} = B_x\, \boldsymbol{dy}{\hat{}}\boldsymbol{dz}$. This machine makes a number by integrating over the surface picked out by the solid arrow lines in the lower right of figure 2.6—the machine asks how many of these oriented tubes cross the surface? Here, there are eighteen of them: $\int_{\text{(surface defined by the arrows)}} \mathrm{F} = 18$. Bong, bong, bong . . .eighteen times.

Slowly, Wheeler et al. build up their theory-machines, one after the other. One crucial stage is the establishment of the covariant

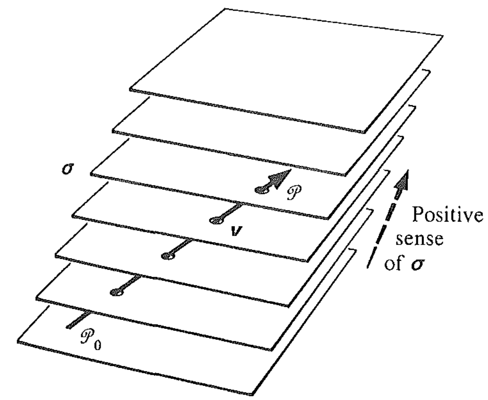

Figure 2.5. Wheeler's 1-form machine. (From Misner, Thorne, and Wheeler, *Gravitation*, figure 2.4, p. 55).

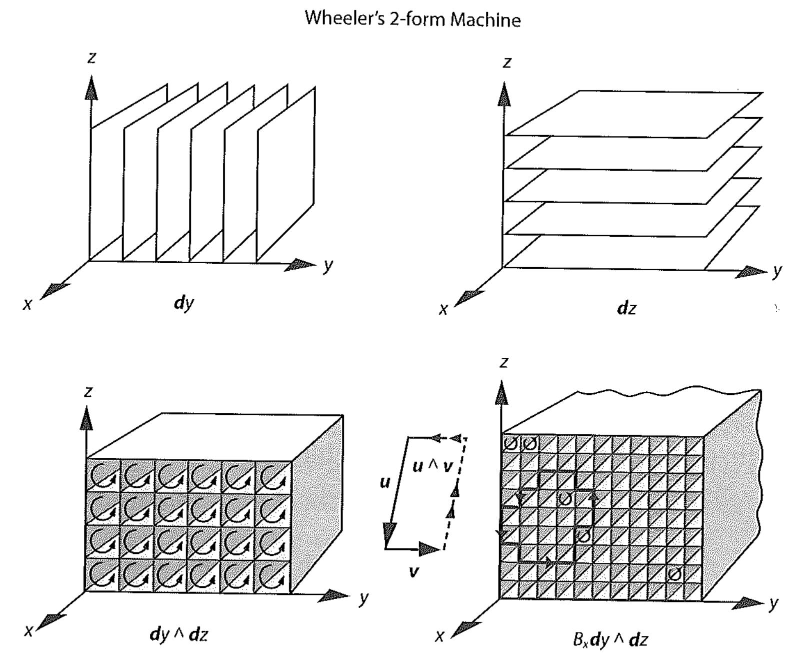

Figure 2.6. Wheeler's 2-form machine. (From Misner, Thorne, and Wheeler, *Gravitation*, figure 4.1, p. 100).

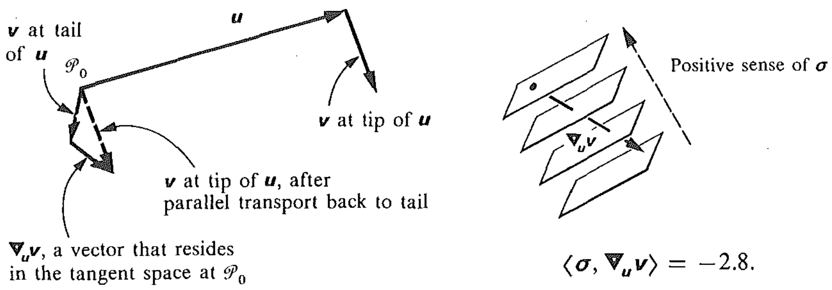

Figure 2.7. Covariant derivative machine (From Misner, Thorne, and Wheeler, *Gravitation*, p. 254).

derivative. *Gravitation:* "Covariant Derivative viewed as Machine: Connection Coefficients as its Components."[34] Like the gravity-chute machines of Franklin Day Jones that take coins, screws, or bullets, this math machine, too, has inputs and outputs:

> the covariant derivative operator [Δ], like most other geometric objects, can be regarded as a machine with slots. There is one such machine at every event P_0 in spacetime.[35]

Δ has three slots. Into the first slot insert a 1-form residing in the tangent space at P_0; into the second slot insert a vector field $v(P)$ defined in neighborhood of P_0; into the third slot insert a vector u that resides in the tangent space at P_0. Presto, the machine spits out a new vector: "the covariant derivative of the vector field v with respect to u."[36] Geometrically, the machine works in two steps: first it transports vector v at the tip of u back to the tail of u. The difference is indicated in figure 2.7 as $\Delta_u v$. Second, the machine counts how many surfaces $\Delta_u v$ pierces of the 1-form σ: here, it seems, 2.8 of them.

Like the machines in the Waltham Watch Company assembly plant, one device's output forms the input to the next. Wheeler takes the covariant derivative machine and uses it to identify the geodesics—the covariant derivative of a geodesic along the geodesic is zero. Then, for every point in space-time, he defines a new machine that takes the deviation of one nearby geodesic from another: this generates yet another machine—the curvature machine. Wheeler's is not only visual

mathematics, it is mathematics read as a series of linked episodic machine stories. One feeds the next: rod moves cam rotates gear. The story unfolds in time under the ever-present sign of the diagram.

4. Theomathematics

Wheeler was always interested in the blasting of matter: from the dynamite cap he detonated as a boy that blew off part of one of his fingers through his work on nuclear weapons; to his examination of positronium (an orbiting electron-positron "hydrogen atom") as it collapsed into pure energy. Indeed, Wheeler wrote a paper just after World War II in which he used Einstein's $E = mc^2$ as what he called "the sextant equation," orienting physicists as to how far they were down the road to the pure and total annihilation of matter.[37] So perhaps it is not too surprising that when Wheeler began to work on general relativity in 1952–53, he chose stellar collapse as one of his prime subjects (emerging from his Los Alamos studies of high-pressure equations of state). Wheeler introduced the term "black hole" in 1968.[38]

He never let go. Years later, Wheeler wrote, "Some day a door will surely open and expose the glittering central mechanism of the world in its beauty and simplicity. Toward the arrival of that day, no development holds out more hope than the paradox of gravitational collapse. Why paradox? Because Einstein's equation says 'this is the end' and physics says 'there is no end.' Why hope? Because among all paradigms for probing a puzzle, physics proffers none with more promise than a paradox."[39]

In order to encompass spin—the necessary building block of elementary particles—one needs to be able to change the connectivity of space. But, as Wheeler recognized full well, classical differentiable geometry refuses to accommodate and—if this doesn't make Bourbaki roll over, nothing will—Wheeler analogizes math to the law. Of litigators, Wheeler sees two types: one type says what you can't do, the second tells you what you have to do and how to do it: "From the first lawyer, classical differential geometry, the client [that is, the physicist] goes away disappointed, still searching."[40] Without a change in connectivity there is neither a way to characterize electric charge as lines of force trapped in the topology of space nor to account for the end moments of gravitational collapse. "Pondering his problems, he comes to the office of a second lawyer, with the name 'Pregeometry' on the door. Full of hope, he knocks and enters."[41]

Now comes another Wheeler story, another machine, each one feeding into the one after it. One thinks here of the famous celestial mechanician and philosopher Pierre Duhem, who related with some horror the contrast between British science (by a Maxwell or a Thompson), which he saw as a squalid factory, and the well-ordered chambers of French physics. Wheeler's mathematical narrative is a bit like Duhem's image, but we can characterize it even further. Wheeler is an episodic narrator, a physicist-author who begins toward the end of a mechanical assembly line and, with each subsequent chapter, brings us closer and closer to the beginning.

Einstein's machine (his general relativity equations) says that "matter tells space how to curve, and space tells matter how to move" (in Wheeler's famous formulation). Then he goes backward, always asking, how did the input for the thought (or chapter n) get produced by the output of chapter $(n - 1)$? Here is an example: to integrate his geometric representation of spin-1/2 particles, he needs space itself to have a fluctuating dimension. How can this be? Why, he asks, does space have the number of dimension that it does? This requires another story.

"Recall the notion of a Borel set," Wheeler says back in 1964. "Loosely speaking, a Borel set is a collection of points ('bucket of dust') which have not yet been assembled into a manifold of any particular dimensionality."[42] Now, quantum mechanics says that there must be amplitudes for the different configurations of anything, so in particular there must be different probability amplitudes for different configurations of Borel sets assembled into structures. This, Wheeler continues, ought to be more likely for lower dimensions (one dimension, two dimensions, three dimensions). But 1D, 2D, and 3D are too uninteresting to produce any useful physics. Four dimensions is interesting—and more likely than five. Wheeler: "Can four therefore, be considered to be the unique dimensionality which is at the same time high enough to give any real physics and yet low enough to have great statistical weight?"[43] Connections arise and vanish between every pair of two points—quantum mechanics says there is no universal answer to the question of what the

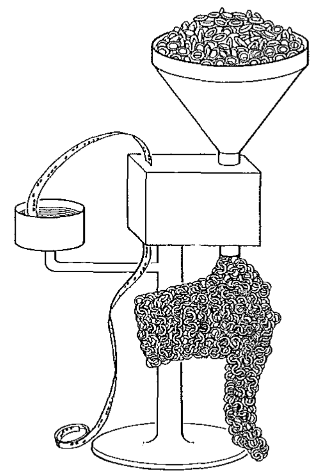

Figure 2.8. The vizier's machine of ten thousand rings. (From Misner, Thorne, and Wheeler, *Gravitation*, figure 44.3, p. 1210).

nearest neighbors of two points are. And so, in this fluctuating world, dimensionality itself disappears.

To get at the quantum construction of dimensional space, Wheeler tells a story (of course) based on an imaginary machine. A vizier gives the following command: put 10,000 rings in the funnel of the machine, and launch a tape with the command to attach one ring to the next. Clatter, clatter, clatter, and a chain 10,000 rings long rattles to the table. Next, a more complex vizier request: the tape tells a more complex series of commands—"this time it is not a one-dimensional structure that emerges, but a two-dimensional one: a Crusader's coat of mail."[44] (See figure 2.8.) Then, the vizier insists on a random tape—this time out of the machine comes a whole series of ornaments, some one-dimensional chains, some two-dimensional structures, some three-dimensional ones. Finally, imagine a more quantum mechanical set of instructions—using the complex numbers quantum mechanics uses for wave functions—to fix which rings attach to which. Wheeler asks, What kind of structures are dominant? What dimensionality prevails most often? What, in short, are the statistical features that emerge as pregeometry is scrambled at the end of each cycle of the universe?

So where does Wheeler's *Gravitation* Bible end? He has gone from matter to fields, from fields to space-time, from a continuum in space-time to an unstable topological quantum foam, to an underlying ontology of Borel set theory, scrambled by quantum mechanics. Since he takes quantum mechanics to lend itself to formulation as a series of logical propositions, the first things of the world are, unexpectedly, propositions. (Of course, never failing to mechanize ideas, Wheeler quickly adds that propositions themselves are, as we know from the days of Shannon, equivalent to switching circuits.) Encoded in his lapidary slogan, "It from bit," Wheeler's world-machine, acknowledged to be woefully incomplete (merely "an idea for an idea"), makes the quantum statistics of propositions into the First Machine, the machine that organizes what the distribution of pregeometries of the world will be like as they emerge from the cyclic big crunch. Wheeler: "Who would have imagined describing something so much a part of the here and now as gravitation in terms of the curvature of the geometry of spacetime? . . . Little astonishment there should be . . . if the description of nature carries one in the end to logic, the ethereal eyrie at the center of mathematics."[45]

Sketchily, speculatively, narrated through an assembly line of idea-machines, Wheeler ends his over-1,200-page book with a schema that in some ways is canonically mathematical to the core:

Logic → Set → Topology → Geometry → Physics . . . .

In an intriguing turn of fate, the Wheeleresque version of mathematical hierarchy has, in his own way and with obvious differences such as the inclusion of logic, paralleled the Bourbakian one. And yet the two mathematical narratives are about as far apart as they could possibly be. Where Bourbaki took the hierarchy of fields as existing out of time, as a logical structure, a logical *architecture*, Wheeler and his geometrodynamic allies rendered it a moving machine where mathematics *was* the physical universe.

5. Bucket of Dust to Bucket of Dust

Bourbaki's world—the world of *Éléments de mathématique*—was fiercely impersonal, voided of heuristics, stripped of images; the collective proudly squelched the individual voice in favor of the group. Theirs was a series of books that was not in the first or even last instance a textbook; more a monument than a story unfolding in time.

Bourbaki's early nomenclature of "abstract package" captured perfectly the impulse driving the project. For this was truly an imagined mathematical architecture that would not dwell unnecessarily on the specifics of a field: no fetishization of this or that corpus of mathematical objects in their specificity. No philosophical musings over the reality status of this or that mathematical object. Instead, Bourbaki was after a generalized *Bauplan*, one that could be applied again and again to different domains. It was precisely in this repetition of form that Taylorism (economy of thought) could meet Haussmannianism (imperial, radial roots to the worker-suburbs of knowledge). Here is imperial knowledge, modernism as the rigorous progression from the general to the specific, from high abstraction to low materiality; from the center of Paris at the École normale supérieure, rue d'Ulm, to the hammering, forging, making of the suburbs.

Asked why there was such a lack of visual illustration in Bourbaki's canonical works, Pierre Cartier responded, "The Bourbaki were Puritans, and Puritans are strongly opposed to pictorial representations of truths of their faith. The number of Protestants and Jews in the Bourbaki group was overwhelming. . . . And then there was the idea that there is an opposition between art and science. Art is fragile and mortal, because it appeals to emotions, to visual meaning, and to unstated analogies."[46]

Wheeler's world was altogether different. Far from an iconoclastic suspicion of the diagrammatic, he loved images, took drawing lessons, studied mechanical illustration in his engineering training, and saw the visual as a crucible in which to test arguments: "I certainly feel that any idea that's reasonable lets itself be depicted in a picture that has some impact. If I can't make a picture, I feel there's something faulty about the idea or the thoroughness with which it's been investigated."[47] "I would be happy if the whole of physics could be expressed in the form of simple attractive diagrams. It's a continual challenge to me to look at the Sistine Chapel painting by Michelangelo of the creation, with the finger of the Lord reaching out toward the figure of man and giving life," to which he added, just a tad immodestly, "I have an equally impressive diagram on how quantum physics takes its origin."[48]

More "physics package" (weapons designer term for the nuclear part of the device) than Bourbakian "abstract package," Wheeler's theme throughout is not so much the stable pyramid of knowledge culminating abstract perfection as it was in a machine-like and often apocalyptic vision of matter.

Fission (1939)
Manhattan Project, A-bomb (1942–45)
Positronium (1946)
Fusion (1949–52)
Vacuum fluctuations (1947–48)
Matterhorn B, H-bomb (1949–52)
Completely collapsed objects—leading to black holes (1960s)
Universal reprocessing (1990s)

The end state fascinated Wheeler: the final collapse of the universe is all the more apocalyptic for happening in cyclic infinity. "Ordinary" gravitational collapse of one or more stars kills all the individuality of these objects—left only are the mass, the charge, and the angular momentum. But the final collapse of a closed universe goes much further. The total charge of the universe as a whole because all lines of force go back to the same charged point. Total mass and angular momentum must be zero because there is no external flat space in which the motion of test masses could give meaning to these concepts. Full, universe-wide collapse cuts down the last tree standing, the final recourse of physical laws in the conservation of charge, lepton and baryon numbers, mass, angular momentum. Gone. Wheeler: "the established is disestablished. No determinant of motion does one see left that could continue unchanged in value from cycle to cycle of the universe." Even the spectrum of particle masses must be lost. Planck, Wheeler recalled, had bequeathed us units given in terms of the characteristic constants—gravitational, black body, and speed of light: mass (10^{-5} g), length (10^{-33} cm), and time (10^{-43} sec). Wheeler expected that these too would be extinguished and reborn with other numbers each time the universe went through its "reprocessing."[49]

Gravitation fastens on, dwells in, the absolute and total annihilation of every single last vestige of order and leaves the reader with the paradox that pits "physics comes to an end" against "physics must go on." And so

when the "Bible" of gravity ends, it does so extolling the crisis that dwarfs even the crisis that had led Bohr to the advent of quantum mechanics back in 1911: "No predictions subject to early test are more entrancing than those on the formation and properties of a black hole, 'laboratory model' for some of what is predicted for the universe itself. No field is more pregnant with the future than gravitational collapse. No more revolutionary views of man and the universe has one ever been driven to consider seriously than those that come out of pondering the paradox of collapse, the greatest crisis of physics of all time."[50]

Perhaps, then, it should not surprise us too much if, as Wheeler approaches the beginning-end of all things, there is a bucket of Borelian dust. Out of this filth, through the proposition machine of quantum mechanics comes pregeometry; pregeometry makes geometry; geometry gives rise to matter and the physical laws and constants of the universe. At once close to and far from the crystalline story that Bourbaki invoked, Wheeler's genesis puts one in mind of Genesis 3:19: "In the sweat of thy face shalt thou eat bread, till thou return unto the ground; for out of it wast thou taken: for dust thou art, and unto dust shalt thou return."[51]

NOTES

1. The founding members of the Bourbaki mathematical collective were Henri Cartan, Claude Chevalley, Jean Delsarte, Jean Dieudonné, René de Possel, and André Weil. They chose their pseudonym, Nicolas Bourbaki, around 1938. Liliane Beaulieu, "A Parisian Café and Ten Proto-Bourbaki Meetings (1934–1935)," *Mathematical Intelligencer* 15 (1993): 27–35. For more on Bourbaki, see, e.g., David Aubin, "The Withering Immortality of Nicolas Bourbaki: A Cultural Connector at the Confluence of Mathematics," *Science in Context* 10 (1997): 297–342, in which the author locates Bourbaki structure as part of a larger cultural move in that direction even if the links were sometimes, as Aubin puts it, superficial. Liliane Beaulieu studies their self-representation in history in her "Bourbaki's Art of Memory," in *Osiris*, 2nd ser., 14 ("Commemorative Practices in Science: Historical Perspectives on the Politics of Collective Memory") (2002): 219–251. Leo Corry's excellent study *Modern Algebra and the Rise of Mathematical Structure* (Basel: Birkhäuser, 1996) analyzes the image of mathematics that lay behind Bourbaki's mix of Platonism, formalism, and the role of its axiomatically oriented hierarchy of structures. The ad hoc nature of the actual deployment of structure is emphasized in Corry's "Nicolas Bourbaki and the Concept of Mathematical Structure," *Synthese* 92, no. 3 (1992): 315–48. Two short secondary works are also helpful, J. Fang, *Bourbaki: Toward a*

Philosophy of Modern Mathematics (Hauppauge, NY: Paideia Press, 1970), which seeks to defend Bourbaki's approach, and Maurice Mashaal, *Bourbaki: A Secret Society of Mathematicians*, trans. Anna Pierrehumbert (Providence, RI: American Mathematical Society, 2006), which offers a lively pedagogical introduction to the mathematics and personalities. Of the many participant histories, Armand Borel's, Henri Cartan's, and J. Dieudonné's are particularly vivid: Borel, "Twenty-Five Years with Nicolas Bourbaki, 1949–1973," *Notices of the ACM* 45, no. 3 (March 1998): 373–80; Cartan, "Nicolas Bourbaki and Contemporary Mathematics," *Mathematical Intelligencer* 2 (1980): 175–80; and Dieudonné, "The Difficult Birth of Mathematical Structures, 1840–1940," in *Scientific Culture in the Contemporary World*, ed. Vittorio Mathieu and Paolo Rossi (Milan: Scientia, 1979).

2. Nicolas Bourbaki, *Éléments de mathématique* (Paris: Hermann, 1939); Charles W. Misner, Kip S. Thorne, and John A. Wheeler, *Gravitation* (San Francisco, CA: W. H. Freeman, 1973).

3. B. L. van der Waerden, *Moderne Algebra* (Berlin: J. Springer, 1930–31).

4. Nicolas Bourbaki, "The Architecture of Mathematics," *American Mathematical Monthly* 57, no. 4 (April 1950): 221–32.

5. Bourbaki, "Architecture," 227.

6. Ibid., 227.

7. Ibid., 225.

8. Robert Osserman, "Structure versus Substance: The Rise and Fall of Geometry," *Two-Year Mathematics Journal* 12, no. 4 (1981): 239–46, quotation at 243.

9. Ibid., 230.

10. Ibid., 231.

11. From "Claude Chevalley Described by His Daughter (1988)," in Michèle Chouchan, *Nicolas Bourbaki: Faits et légendes* (Paris: Éditions du Choix, 1995), 36–40, translated and cited in Marjorie Senechal, "The Continuing Silence of Bourbaki: An Interview with Pierre Cartier, June 18, 1997," *Mathematical Intelligencer* 1 (1998): 22–28, http://www.ega-math.narod.ru/Bbaki/Cartier.html.

12. Quotation from report on the first meeting, translation and citation from Liliane Beaulieu, "Dispelling a Myth: Questions and Answers about Bourbaki's Early Work, 1934–1944," in *The Intersection of History and Mathematics*, ed. Sasaki Chikara, Sugiura Mitsuo, and Joseph W. Dauben (Basel: Birkhäuser, 1994), 244–45 (typescript pagination).

13. Interview of John Wheeler by Charles Weiner and Gloria Lubkin on April 5, 1967, Niels Bohr Library and Archives, American Institute of Physics, College Park, MD, http://www.aip.org/history/ohilist/4958.html. Hereafter AIP 1967.

14. Ibid.

15. Ibid.

16. Ibid.

17. Ibid.

18. Franklin D. Jones, *Mechanisms and Mechanical Movements: A Treatise on Different Types of Mechanisms and Various Methods of Transmitting, Controlling and Modifying Motion, To Secure Changes of Velocity, Direction, and Duration or Time of Action* (New York and London: Industrial Press, 1920). Reprinted as Franklin D. Jones, ed., *Ingenious Mechanisms for Designers and Inventors* (New York: Industrial Press, 1930).

19. On Maxwell's classification of machines, see Peter Galison, *How Experiments End* (Chicago: University of Chicago Press, 1987), chap. 2, 21–75.

20. Jones, *Mechanisms and Mechanical Movements*, v.

21. Ibid., v–vi.

22. Ibid., v–vi.

23. Ibid., 291, figure on 292.

24. Wheeler, interview, AIP 1967.

25. Peter Galison, *Image and Logic* (Chicago: University of Chicago Press, 1997), chap. 4, 239–303.

26. Wheeler, interview, AIP 1967.

27. Wheeler to Feynman, March 29, 1951, Feynman Papers, RPF 3.10, Caltech Archives, Pasadena, CA.

28. Feynman to Wheeler, April 5, 1951, ibid.

29. Wheeler notebook, Relativity I, November 12, 1952, 24, Wheeler Papers, American Philosophical Society Library, Philadelphia.

30. Wheeler QED Notebook I, October 11, 1953, 19, ibid.

31. Wheeler, interview, AIP 1967.

32. Wheeler to Feynman, April 27, 1966, corrected by hand to May 9, 1966 ("9 May 1966"), emphasis added, Feynman Papers, RPF file 3.10, Caltech Archives, Pasadena, CA.

33. Misner, Thorne, and Wheeler, *Gravitation*, 115.

34. Ibid., 254.

35. Ibid., 254.

36. Ibid., 254–56.

37. For a discussion of Wheeler's gloss of Einstein's "sextant equation," see Galison, "The Sextant Equation: $E = mc^2$," in Graham Farmelo, *It Must be Beautiful: Great Equations of Modern Science* (London & New York: Granta, 2002), 68–86.

38. Interview of John A. Wheeler by Kenneth W. Ford on December 6, 1993, Niels Bohr Library and Archives, American Institute of Physics, College Park, MD, http://www.aip.org/history/ohilist/5908_1.html.

39. Misner, Thorne, and Wheeler, *Gravitation*, 1197.

40. Ibid., 1203.

41. Ibid., 1203.

42. Excerpt from Wheeler, "Geometrodynamics and the Issue of the Final State," in DeWitt, C. and B. S. DeWitt, eds., *Relativity, Groups, and Topology* (New York: Gordon and Breach, 1973). Reprinted in Misner, Thorne, and Wheeler, *Gravitation*, 1205.

43. Ibid.

44. Ibid., 1210.

45. Ibid., 1212.
46. Senechal, "The Continuing Silence of Bourbaki," http://www.ega-math.narod.ru/Bbaki/Cartier.html.
47. Interview of John Wheeler by Kenneth W. Ford on March 28, 1994, Niels Bohr Library and Archives, American Institute of Physics, College Park, MD, www.aip.org/history/ohilist/5908_12.html.
48. Interview of John Wheeler by Kenneth W. Ford on March 28, 1994, Niels Bohr Library and Archives, American Institute of Physics, College Park, MD, www.aip.org/history/ohilist/5908_12.html.
49. Misner, Thorne, and Wheeler, *Gravitation*, 1215.
50. Ibid., 1217.
51. *The Holy Bible*, King James Version (New York: Viking Studio, 1999).

CHAPTER 3

Deductive Narrative and the Epistemological Function of Belief in Mathematics

On Bombelli and Imaginary Numbers

FEDERICA LA NAVE

The story of a mathematical discovery is often presented as a linear succession of events corresponding to a series of logical steps leading up to the moment of discovery by proof. The discovery itself takes on the character of a "truth revelation." Such an accounting is cathartic. It makes us feel good about ourselves; it gives us confidence in the power of our mind. But is a sequence of logical steps all there is behind proving something in mathematics? When telling a story, one naturally lapses into a linear mode. But when trying to locate the history of a discovery, we should be prepared for emerging bits and pieces to coalesce into a narrative frame that is not necessarily built on linear deduction.

Narrating the story of a discovery as a linear process, one that moves from intuition to deductive certainty, risks obscuring important pieces of the thought process in mathematics. One of the pieces likely to be lost is the role of belief in proving mathematical propositions. In the course of thinking about and proving mathematical propositions, a mathematician's belief changes. Understanding the complex interactions of the factors influencing such changes in belief is critical to developing a more complete notion of what is involved in proving in mathematics.

I approach the issue of changes in belief by considering a particular historical case, Rafael Bombelli and his struggles to believe in the existence of what he described as a new kind of number (which we call *imaginary numbers*). Bombelli was the first mathematician to accept